A Headache-causing Problem

Conway J.H., Paterson M.S., Moscow U.S.S.R. (1977)

David McQuillan March WU3A 2025

The Conway-Paterson-Moscow Theorem

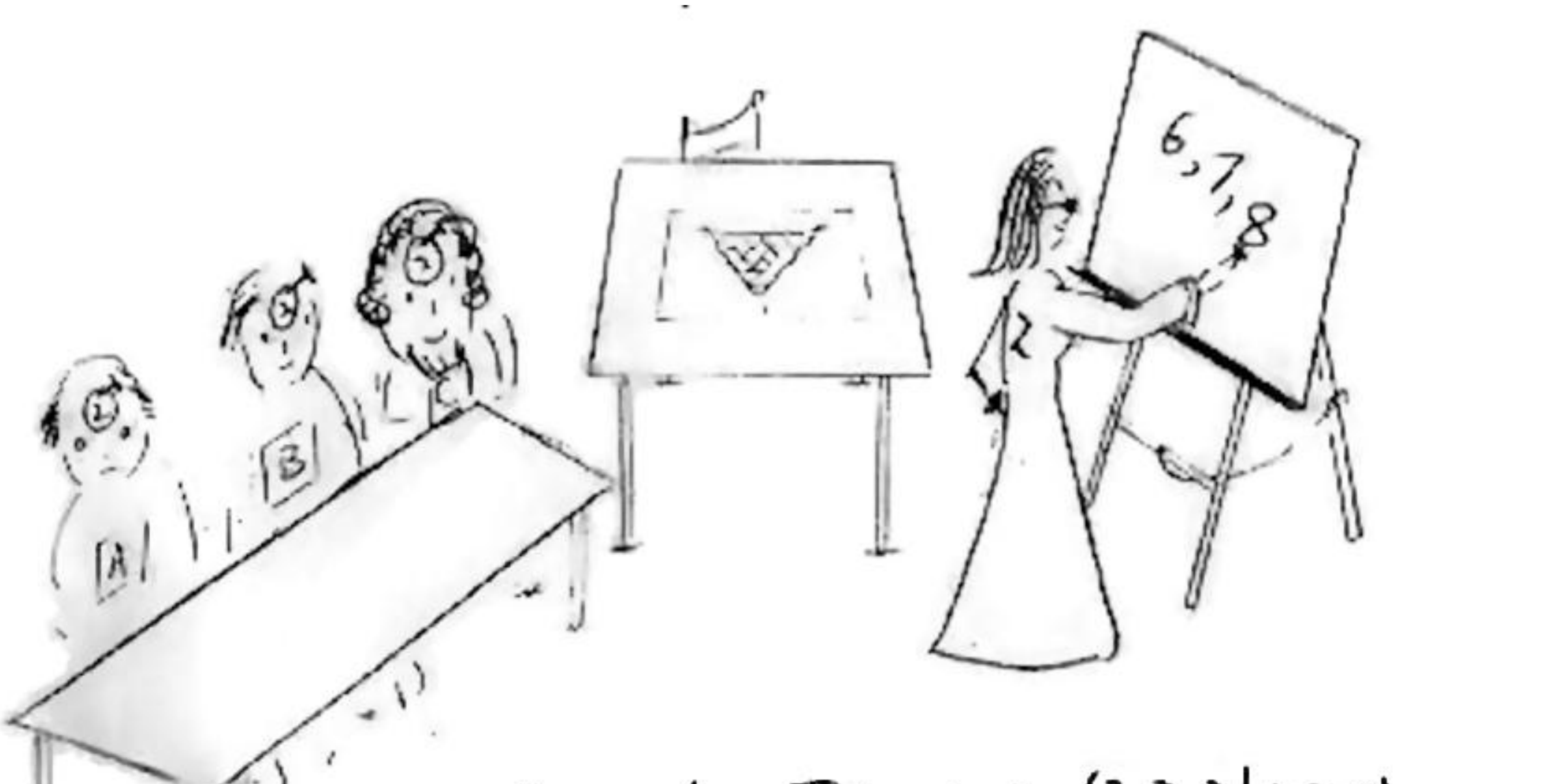

Figure 1 The game (2,2,2|6,7,8)

Statement of Theorem

- Non-negative integers are written on the blackboard and on the discs on the men's foreheads
- The men can see all the numbers except the one on their own forehead and can't tell each other what their numbers are.
- The sum of the numbers on the men's foreheads is equal to one of the numbers on the board.
- Then if the number of men is greater then or equal to the number of numbers on the board, one will eventually be able to deduce the number on their forehead after each is asked in turn cycling round the circle if they have deduced the number on their forehead.

An Example

The cartoon shows an example where three men, Arthur, Bertram and Charles have had discs with the number two put on their foreheads by the janitor acting on the instructions of Zoe at the blackboard. She's blind but she's written 6, 7 and 8 on the board. (2,2,2 | 6, 7, 8) is a way of describing the game.

They are all infinitely intelligent and honourable. They would like to go off to lunch but they like Zoe and are willing to humour her when she makes up games like this, and she has assured them it wouldn't go on forever, even if Duncan and Englebert were there and five numbers were on the board.

Solving (2,2,2 | 6,7,8)

Arthur and Bertram will be asked first but let's what Charles thinks.

I can see two 2's so I must have 2, 3 or 4 so they add up to 6, 7 or 8.

If I have a 2 then Arthur will see two 2's and conclude he must have 2, 3, or 4 and say No – he doesn't know what's on his forehead. Bertram will see two 2's and conclude using the same argument he has a 2, 3 or 4 but must say No.

If I have a 3 then Arthur will conclude he has 1, 2 or 3 and must say No. Bertram will see a 2 and a 3 and also say No.

And if I have a 4 Arthur will conclude he has 0, 1 or 2 and say No. Bertram will see 2 and 4 and also say No.

So I can't decide between 2, 3, and 4 so I say No.

Solving continued – or perhaps not!

So all three say No. And they all knew this was going to happen. And they start the cycle again and Arthur is asked again, then Bertram, then Charles..., and they answer No – No – No... again.

Charles concludes it is most likely that all three of them will each intone No a half dozen times and he still won't know what his number is - but by then they'll all be happy to just end the game and go off and have lunch.

… and actually as we'll see, Charles is quite correct.

So that's the theorem in tatters isn't it?

Zoe the Blind Umpire

Zoe doesn't know what numbers are on the peoples heads, only the numbers on the board. So what can she do as they say No? Well she can write down all the possible situations to start with:-

[0, 0, 6] [0, 0, 7] [0, 0, 8] [0, 1, 5] [0, 1, 6] [0, 1, 7] [0, 2, 4] [0, 2, 5] [0, 2, 6] [0, 3, 3] [0, 3, 4]
[0, 3, 5] [0, 4, 2] [0, 4, 3] [0, 4, 4] [0, 5, 1] [0, 5, 2] [0, 5, 3] [0, 6, 0] [0, 6, 1] [0, 6, 2] [0, 7, 0]
[0, 7, 1] [0, 8, 0] [1, 0, 5] [1, 0, 6] [1, 0, 7] [1, 1, 4] [1, 1, 5] [1, 1, 6] [1, 2, 3] [1, 2, 4] [1, 2, 5]
[1, 3, 2] [1, 3, 3] [1, 3, 4] [1, 4, 1] [1, 4, 2] [1, 4, 3] [1, 5, 0] [1, 5, 1] [1, 5, 2] [1, 6, 0] [1, 6, 1]
[1, 7, 0] [2, 0, 4] [2, 0, 5] [2, 0, 6] [2, 1, 3] [2, 1, 4] [2, 1, 5] [2, 2, 2] [2, 2, 3] [2, 2, 4] [2, 3, 1]
[2, 3, 2] [2, 3, 3] [2, 4, 0] [2, 4, 1] [2, 4, 2] [2, 5, 0] [2, 5, 1] [2, 6, 0] [3, 0, 3] [3, 0, 4] [3, 0, 5]
[3, 1, 2] [3, 1, 3] [3, 1, 4] [3, 2, 1] [3, 2, 2] [3, 2, 3] [3, 3, 0] [3, 3, 1] [3, 3, 2] [3, 4, 0] [3, 4, 1]
[3, 5, 0] [4, 0, 2] [4, 0, 3] [4, 0, 4] [4, 1, 1] [4, 1, 2] [4, 1, 3] [4, 2, 0] [4, 2, 1] [4, 2, 2] [4, 3, 0]
[4, 3, 1] [4, 4, 0] [5, 0, 1] [5, 0, 2] [5, 0, 3] [5, 1, 0] [5, 1, 1] [5, 1, 2] [5, 2, 0] [5, 2, 1] [5, 3, 0]
[6, 0, 0] [6, 0, 1] [6, 0, 2] [6, 1, 0] [6, 1, 1] [6, 2, 0] [7, 0, 0] [7, 0, 1] [7, 1, 0] [8, 0, 0]

Zoe figuring it out

Zoe will strike out every situation as soon as she knows the game would have terminated at that point with a Yes.

Arthur says No so Zoe can strike out

[0, 0, 8] [0, 1, 7] [0, 2, 6] [0, 3, 5] [0, 4, 4]

[0, 5, 3] [0, 6, 2] [0, 7, 1] [0, 8, 0]

Because the sum of the other two is 8 so he would have 0 on his forehead. However he wouldn’t say Yes for [0, 0, 7] because [1, 0, 7]

would also be a possibility.

The rule Zoe uses

If A sees b, c, d... And there's only possible value for ? In [?, b, c, d]

Then A can say Yes. Otherwise they will say No.

Bertram's No means Zoe can eliminate

[0, 0, 7] [1, 0, 7] [2, 0, 6] [3, 0, 5] [4, 0, 4]

[5, 0, 3] [6, 0, 2] [7, 0, 1] [8, 0, 0]

The [0, 0, 7] is the only [0, ?, 7] - Arthur eliminated [0, 1, 7]

Continuing

Next Charles says No and that eliminates
[0, 0, 6] [0, 7, 0] [1, 7, 0] [2, 6, 0] [3, 5, 0]
[4, 4, 0] [5, 3, 0] [6, 2, 0] [7, 0, 0] [7, 1, 0]

[0, 7, 0] is eliminated the same as [0, 0, 7] was for Bertram,
but why [0, 0, 6]?

Well Arthur’s No eliminated [0, 0, 8]
and Bertram’s No eliminated [0, 0, 7].

The Theorem again

If the number of numbers on the board is less than or equal to the number of people with discs on their foreheads, then eventually one of them will say Yes (if they're infinitely intelligent and persistent!)

In fact after 18 No's Arthur will say Yes, he has a 2 on his forehead.

The possibilities according to Zoe when Arthur says Yes are

[2, 2, 2] [2, 2, 3] [2, 3, 2] [2, 3, 3]

So after they said No a half dozen times each and then Arthur says Yes Bertram and Charles still don't know if it was a 2 or a 3 on their foreheads. Charles was quite correct!

The Proof start

1. If there is only one person, N=1, then he knows his number if there is one number on the board but will be unable to decide if there is more than one. The proof will be by induction on N.
2. For any set of non-negative numbers on the board there are only a finite set of non-negative numbers the people can have on their foreheads that add up to the numbers on the board.
3. If the No's go on endlessly there must come a time when no more possibilities are removed

Proof Continued

4. Suppose (a,b,c,d,e,...,n) is one of these possibilities that refuses to be eliminated for persons (A,B,C,D,E,...,N). Then for A there must be more than one possible value of a – say a_0 and for the lowest.
5. Consider the situations with a_0 fixed. We can remove it and subtract a_0 from the numbers on the board – only ones still >= 0 can be part of possible sums.
6. Then we have by induction hypothesis that N-1 people B....N need at least (N-1)+1 totals by induction to keep the possibilities endlessy.

Proof completed

7. Adding a_0 back there are at least N totals for **(**a_0 **,**b,c,d,e,...,n)
8. Find the highest possible total for that situation.
9. Then there is an $a_H > a_0$ which when put in will be a possibility and gives a higher total than any of the ones needed before.
10. Therefore at least ((N-1)+1)+1 = N+1 totals would be needed for a game with N people to never end.

A Question I’ve got horribly lost on

So that’s it, but how many No’s are actually required if they can skip over some people if they were going to say No, under what conditions would it be okay to do that?

For instance is Arthur has 2 on his forehead and Bertram has 1 and the sum is 3 or 4. Then both know that using Zoe’s method only [0, 4] is eliminated in the first round and the both know that and that the other knows that and that it isn’t a possibility. The same for Bertram then saying No eliminating [0, 3] and [4, 0]

Another headache-causing problem!?